Hyperpolygon spaces and their cores

Megumi Harada
Department of Mathematics, University of Toronto, Ontario M5S 3G3 Canada.

Nicholas Proudfoot
Department of Mathematics, University of California, Berkeley, CA 94720.

Abstract. Given an n-tuple of positive real numbers $(\alpha_1, \ldots, \alpha_n)$, Konno [K2] defines the *hyperpolygon space* $X(\alpha)$, a hyperkähler analogue of the Kähler variety $M(\alpha)$ parametrizing polygons in $\mathbb{R}^3$ with edge lengths $(\alpha_1, \ldots, \alpha_n)$. The polygon space $M(\alpha)$ can be interpreted as the moduli space of stable representations of a certain quiver with fixed dimension vector; from this point of view, $X(\alpha)$ is the hyperkähler quiver variety defined by Nakajima [N1, N2]. A quiver variety admits a natural $\mathbb{C}^*$-action, and the union of the precompact orbits is called the *core*. We study the components of the core of $X(\alpha)$, interpreting each one as a moduli space of pairs of polygons in $\mathbb{R}^3$ with certain properties. Konno gives a presentation of the cohomology ring of $X(\alpha)$; we extend this result by computing the $\mathbb{C}^*$-equivariant cohomology ring, as well as the ordinary and equivariant cohomology rings of the core components.

Let K be a compact Lie group acting linearly on $\mathbb{C}^N$ with moment map $\mu : \mathbb{C}^N \to \mathfrak{k}^*$ such that $\mu(0) = 0$. Then for any central regular value $\alpha \in \mathfrak{k}^*$, the Kähler quotient

$$M(\alpha) = \mathbb{C}^N /\!\!/_\alpha K = \mu^{-1}(\alpha)/K$$

is a Kähler manifold of complex dimension $N - \dim K$. The cotangent bundle $T^*\mathbb{C}^N$ is a hyperkähler manifold, and the induced action of K on $T^*\mathbb{C}^N$ is hyperhamiltonian (see, for example, [HP]), with hyperkähler moment map

$$\mu_\mathbb{R} \oplus \mu_\mathbb{C} : T^*\mathbb{C}^N \to \mathfrak{k}^* \oplus \mathfrak{k}^*_\mathbb{C}.$$

We call the hyperkähler reduction

$$X(\alpha) = T^*\mathbb{C}^N /\!\!/\!\!/\!\!/_{(\alpha,0)} K = \left(\mu_\mathbb{R}^{-1}(\alpha) \cap \mu_\mathbb{C}^{-1}(0) \right) \Big/ K$$

the *hyperkähler analogue* of $X(\alpha)$. The manifold $X(\alpha)$ is a noncompact hyperkähler manifold of complex dimension $2(N - \dim K)$, containing the cotangent bundle to $M(\alpha)$ as an open set [HP]. The action of the nonzero complex numbers $\mathbb{C}^*$ on $T^*\mathbb{C}^N$ given by scalar multiplication on each fiber induces an action of $\mathbb{C}^*$ on $X(\alpha)$, which restricts to the scalar action on the fibers of $T^*M(\alpha) \subseteq X(\alpha)$. The *core* $\mathfrak{L}$ of $X(\alpha)$ is defined to be the set of points $x \in X(\alpha)$ such that the limit $\lim_{\lambda \to \infty} \lambda \cdot x$ exists (the opposite limit $\lim_{\lambda \to 0} \lambda \cdot x$ always exists). The action of $\mathbb{C}^*$ defines a deformation retraction of $X(\alpha)$ onto $\mathfrak{L}$. If $M(\alpha)$ is nonempty and compact, then $\mathfrak{L}$ is simply the union of all those Białyniski-Birula strata whose closures are compact.

In [HP] we studied the hyperkähler analogues of toric varieties, which arise when K is abelian (see also [BD, HS, K1]). In this case, $\mathfrak{L}$ is a union of toric varieties, one of which is the original toric variety $M(\alpha)$. Another important context in which Kähler reductions and their hyperkähler analogues arise is the study of varieties associated to quivers; this includes the spaces that we will study in this paper.

Suppose given a quiver Q (a directed graph) with vertex set I, and let $\{V_i \mid i \in I\}$ be a collection of finite complex dimensional vector spaces. A *representation* of Q is a collection of maps from V_i to V_j for every pair of vertices i and j connected by an edge. The group $PU(V) = \left(\prod GL(V_i) \right) \big/ GL(1)_\Delta$ acts hamiltonianly on the space $E(Q, V)$ of representations of Q. Both the Kähler quotients $M(\alpha) = E(Q, V) /\!\!/_\alpha PU(V)$ and their hyperkähler analogues $X(\alpha) = T^* E(Q, V) /\!\!/\!\!/\!\!/_{(\alpha, 0)} PU(V)$, called *quiver varieties*, have been studied extensively. A good introduction to these varieties, both the Kähler and hyperkähler versions, can be found in [N2]. Cores of quiver varieties have attracted particular attention in representation theory. The fundamental classes of their components provide a natural basis for the top homology of $X(\alpha)$, which leads to the construction of canonical bases for representations of the modified universal enveloping algebra associated to the quiver Q [N3] (see also [N4] for more recent results along these lines).

The examples with which we will be concerned in this paper are quiver varieties corresponding to a very special class of quivers, as shown in the following picture. We label the
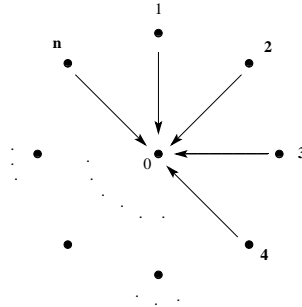

Figure 1: The quiver for hyperpolygon spaces.

vertices 0 through n, with zero in the center, and for each $i \in \{1, \ldots, n\}$ we have an arrow from i to 0. We put $V_i = \mathbb{C}$ for all $i \in \{1, \ldots, n\}$, and $V_0 = \mathbb{C}^2$, so that

$$E(Q, V) = \bigoplus_{i=1}^{n} \operatorname{Hom}(V_i, V_0) \cong \mathbb{C}^{2n}.$$

In this example, the Kähler quiver variety $M(\alpha)$ has a nice geometric interpretation. The group $PU(V)$ is isomorphic to $\left(SU(2) \times U(1)^n \right) \big/ \mathbb{Z}_2$ where $\mathbb{Z}_2$ acts diagonally on the $n+1$ factors, and a central element $\alpha = 0 \oplus (\alpha_1, \ldots, \alpha_n) \in \mathfrak{su}(2)^* \oplus (\mathfrak{u}(1)^*)^n$ is given by an n-tuple of real numbers. The variety $M(\alpha)$ is diffeomorphic to the moduli space of n-sided polygons in $\mathbb{R}^3$, with edge lengths $(\alpha_1, \ldots, \alpha_n)$, modulo the action of $SO(3)$ on $\mathbb{R}^3$ by rotation
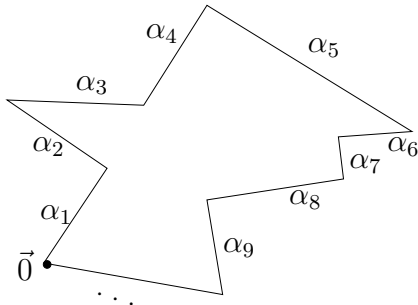
[HK1, HK2, Kl], as in Figure 2.

Figure 2: A polygon in $\mathbb{R}^3$ with specified edge lengths α_i.

The hyperkähler analogue $X(\alpha)$ was introduced in [K1], in which Konno enumerated the components of the core of $X(\alpha)$ (Theorem 2.1), showing that the components other than $M(\alpha)$ are in bijection with the collection of subsets $S \subseteq \{1, \ldots, n\}$ of size at least 2 such that $\sum_{i \in S} \alpha_i < \sum_{j \in S^c} \alpha_j$. Such a subset S will be called *short*. Our first set of results, comprising Section 2, concerns the core components $\{U_S\}$, which one may think of as *generalized polygon spaces*. We prove that U_S is smooth for each short subset S (Theorem 2.2), and interpret it as the moduli space of pairs of polygons in $\mathbb{R}^3$ with certain geometric properties (Theorem 2.5). This is therefore a solution, in the special case of polygon spaces, to the following general problem.

Problem 1 Given any moduli space M that can be constructed as a Kähler reduction (or GIT quotient) of complex affine space, is it possible to interpret the core components of the hyperkähler analogue X as moduli spaces corresponding to other, related moduli problems?

Our second set of results, comprising Sections 3 and 4, concerns the S^1-equivariant cohomology of $X(\alpha)$ and U_S, where $S^1 \subseteq \mathbb{C}^*$ is the unit circle.[1] Konno computes the cohomology of $X(\alpha)$, showing in particular that the hyperkähler Kirwan map

$$\kappa : H_K^*(T^*\mathbb{C}^{2n}) \to H_K^*\left(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\right) \cong H^*(X(\alpha))$$

in the case of hyperpolygon spaces is surjective (Theorem 3.1). We generalize this result by computing the kernel of the *equivariant* Kirwan map

$$\kappa_{S^1} : H_{K \times S^1}^*(T^*\mathbb{C}^{2n}) \to H_{K \times S^1}^*\left(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\right) \cong H_{S^1}^*(X(\alpha))$$

---

[1]The S^1-equivariant cohomology is identical to the $\mathbb{C}^*$-equivariant cohomology; we use S^1 in Section 3 because at times it is convenient to work with the real ADHM description of $X(\alpha)$, in which context the full $\mathbb{C}^*$-action is difficult to write down explicitly.

(Theorem 3.2), which is also surjective by Corollary 3.6. In Section 4, we compute the ordinary and equivariant cohomology of a core component U_S (Theorem 4.1 and Corollary 4.2). All cohomology rings are taken with coefficients in $\mathbb{Q}$.

In general, the compactness of the fixed point set $X(\alpha)^{S^1}$ gives equivariant cohomology many nice properties not shared by the ordinary cohomology of $X(\alpha)$; for example, the localization theorem of [AB] makes possible a theory of integration in equivariant cohomology, provided that the fixed point set is compact. Nakajima studies the S^1-equivariant cohomology and K-theory of quiver varieties in [N4], using it to construct representations of quantum affine algebras associated to Q. In the process, he conjectures that the equivariant Kirwan map is surjective for all quiver varieties [N4, 7.5.1].

The reader is advised that cohomology computations in hyperpolygon spaces and generalized polygon spaces are often motivated by beautiful and intuitive geometry, but they are just as often driven by daunting, labyrinthine algebra. Whenever possible, we precede the proofs of our theorems with remarks that are aimed at making the geometry maximally transparent.

1 Hyperpolygon spaces

We begin by collecting the basic definition and properties of a hyperpolygon space, most of which can be found in [K2]. Fix a positive integer $n \geq 3$, and consider the group[2]

$$G := \Big(SL(2, \mathbb{C}) \times (\mathbb{C}^*)^n \Big) \Big/ \mathbb{Z}_2,$$

where $\mathbb{Z}_2$ acts by multiplying each factor by -1. We define a right action of G on $\mathbb{C}^{2n}$ as follows. We will write an element of $\mathbb{C}^{2n}$ as an n-tuple of column vectors

$$q = (q_1, \ldots, q_n),$$

and put

$$q[A; e_1, \ldots, e_n] = (A^{-1} q_1 e_1, \ldots, A^{-1} q_n e_n).$$

The compact subgroup

$$K := \Big(SU(2) \times U(1)^n \Big) \Big/ \mathbb{Z}_2 \subseteq G$$

---

[2]One may prefer to just consider the group $SL(2, \mathbb{C}) \times (\mathbb{C}^*)^n$, and allow it to act with a finite kernel. We quotient by $\mathbb{Z}_2$ only to be consistent with the conventions of [N2] and [K2].

acts with moment map $\mu : \mathbb{C}^{2n} \to \mathfrak{su}(2)^* \oplus (\mathfrak{u}(1)^*)^n$ given by the equation

$$\mu(q) = \sum_{i=1}^{n} (q_i q_i^*)_0 \oplus \left(\frac{1}{2}|q_1|^2, \dots, \frac{1}{2}|q_n|^2 \right),$$

where q_i^* denotes the conjugate transpose of q_i, $(q_i q_i^*)_0$ denotes the traceless part of $q_i q_i^*$, and $\mathfrak{su}(2)^*$ is identified with $i \cdot \mathfrak{su}(2)$ via the trace form. Given an n-tuple of real numbers $(\alpha_1, \dots, \alpha_n)$, we define the *polygon space*

$$M(\alpha) := \mathbb{C}^{2n} /\!\!/_\alpha K = \mu^{-1}(\alpha)/K,$$

where $\alpha = 0 \oplus (\alpha_1, \dots, \alpha_n) \in \mathfrak{su}(2)^* \oplus (\mathfrak{u}(1)^*)^n$. If we break the reduction into two steps, reducing first by $U(1)^n$ and then by $SU(2)$, we find that

$$M(\alpha) \cong \left\{ (v_1, \dots, v_n) \in (\mathbb{R}^3)^n \, \middle| \, \|v_i\| = \alpha_i \text{ and } \sum v_i = 0 \right\} \Big/ SO(3) \qquad (1)$$

(see Remark 2.6 and the proof of Theorem 2.5). Here $\mathfrak{su}(2)^*$ is being identified with $\mathbb{R}^3$, and the coadjoint action of $SU(2)$ on $\mathfrak{su}(2)^*$ is being replaced by the standard action of $SO(3)$ on $\mathbb{R}^3$ [HK2]. This space, therefore, may be thought of as the moduli space of n-sided polygons in $\mathbb{R}^3$, with fixed edge lengths, up to rotation. In particular, $M(\alpha)$ is empty unless $\alpha_i \geq 0$ for all i.

We call α *generic* if there does not exist a subset $S \subseteq \{1, \dots, n\}$ such that $\sum_{i \in S} \alpha_i = \sum_{j \in S^c} \alpha_j$. Geometrically, this means that there is no element of $M(\alpha)$ represented by a polygon that is contained in a single line in $\mathbb{R}^3$. If α is generic, then $M(\alpha)$ is smooth [HK1]. Throughout this paper we will assume that α is generic, and that $\alpha_i > 0$ for all i.

To define the hyperkähler analogue of $M(\alpha)$, we consider the induced action of G on $T^*\mathbb{C}^{2n}$. Explicitly, we write an element of $T^*\mathbb{C}^{2n}$ as (p, q), where $q = (q_1, \dots, q_n)$ is an n-tuple of column vectors and $p = (p_1, \dots, p_n)$ an n-tuple of row vectors, and we put

$$(p, q)[A; e_1, \dots, e_n] = \left((e_1^{-1} p_1 A, \dots, e_n^{-1} p_n A), (A^{-1} q_1 e_1, \dots, A^{-1} q_n e_n) \right).$$

The vector space $T^*\mathbb{C}^{2n}$ has the structure of a hyperkähler manifold, and the action of K on $T^*\mathbb{C}^{2n}$ is hyperhamiltonian with hyperkähler moment map [K2] (see also [HP])

$$\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : T^*\mathbb{C}^{2n} \to \left(\mathfrak{su}(2)^* \oplus (\mathfrak{u}(1)^*)^n \right) \oplus \left(\mathfrak{sl}(2, \mathbb{C})^* \oplus (\mathfrak{u}(1)_{\mathbb{C}}^n)^* \right)$$

given by the equations

$$\mu_{\mathbb{R}}(p, q) = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} (q_i q_i^* - p_i^* p_i)_0 \oplus \left(\frac{1}{2} \left(|q_1|^2 - |p_1|^2 \right), \dots, \frac{1}{2} \left(|q_n|^2 - |p_n|^2 \right) \right)$$

and

$$\mu_{\mathbb{C}}(p,q) = -\sum_{i=1}^{n}(q_i p_i)_0 \oplus \left(\sqrt{-1}p_1 q_1, \ldots, \sqrt{-1}p_n q_n\right).$$

We then define the *hyperpolygon space* to be the hyperkähler quotient

$$X(\alpha) := T^*\mathbb{C}^{2n} /\!/\!/\!/_{(\alpha,0)} K = \left(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\right)\Big/ K,$$

a smooth, noncompact hyperkähler manifold of complex dimension $2(n-3)$ [K2].

The polygon and hyperpolygon spaces $M(\alpha)$ and $X(\alpha)$ are precisely the Kähler and hyperkähler varieties associated by Nakajima to the quiver shown in Figure 1. It is shown in [N1] that

$$M(\alpha) \cong \left(\mathbb{C}^{2n}\right)^{\alpha-\mathrm{st}}/G \qquad \text{and} \qquad X(\alpha) \cong \mu_{\mathbb{C}}^{-1}(0)^{\alpha-\mathrm{st}}/G,$$

where α-st means stable with respect to the weight α in the sense of geometric invariant theory. Nakajima gives a stability criterion for general quiver varieties [N1, N2], which Konno interprets in the special case of hyperpolygon spaces. Call a subset $S \subseteq \{1, \ldots, n\}$ *short* if $\sum_{i \in S} \alpha_i < \sum_{j \in S^c} \alpha_j$, otherwise call it *long*. (Assuming that α is generic is equivalent to assuming that every subset is either short or long.) Given a point $(p,q) \in T^*\mathbb{C}^{2n}$ and a subset $S \subseteq \{1, \ldots, n\}$, we will say that S is *straight* in (p,q) if q_i is proportional to q_j for every $i,j \in S$. The terminology comes from Kähler polygon spaces, in which this condition is equivalent to asking that the vectors v_i and v_j be proportional over $\mathbb{R}_+$, or that the edges of lengths α_i and α_j (if they happen to be adjacent) line up to make a single edge of length $\alpha_i + \alpha_j$, as in Figure 3.
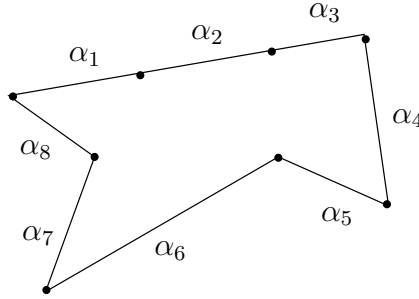

Figure 3: The subset $\{1, 2, 3\}$ is straight.

Theorem 1.1 [K2, 4.2] *Suppose that α is generic, and $\alpha_i > 0$ for all i. Then a point $(p,q) \in T^*\mathbb{C}^{2n}$ is α-stable if and only if the following two conditions are satisfied:*

    1) $q_i \neq 0$ *for all i, and*

    2) *if S is straight and $p_j = 0$ for all $j \in S^c$, then S is short.*

We will use the notation $[p, q]$ to denote the G-equivalence class of a point $(p, q) \in \mu_{\mathbb{C}}^{-1}(0)^{\alpha-\mathrm{st}}$, and $[p, q]_{\mathbb{R}}$ to denote the K-equivalence class of a point $(p, q) \in \mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)$. Note that $M(\alpha)$ sits inside of $X(\alpha)$ as the locus of points $[p, q]$ with $p = 0$. This observation, along with Theorem 1.1, allows us to recover the α-stability condition for the action of G on $\mathbb{C}^{2n}$. A point $q \in \mathbb{C}^{2n}$ is α-stable if and only if $q_i \neq 0$ for all i, and no long subset is straight, as first shown in [Kl]. The polygonally-minded reader is warned that in the hyperpolygon space $X(\alpha)$, long subsets *can* be straight.

2 The core

For the rest of the paper we fix a generic $\alpha = 0 \oplus (\alpha_1, \ldots, \alpha_n) \in \mathfrak{su}(2)^* \oplus (\mathfrak{u}(1)^*)^n$, with $\alpha_i > 0$ for all i. To simplify notation, we will write $X = X(\alpha)$ and $M = M(\alpha)$. Consider the action of $\mathbb{C}^*$ on X given in the complex description by $\lambda \cdot [p, q] = [\lambda p, q]$. The circle $S^1 \subseteq \mathbb{C}^*$ preserves $\omega_{\mathbb{R}}$, and acts with moment map $\Phi : X \to \mathbb{R}$ given in the symplectic description by $\Phi([p, q]_{\mathbb{R}}) = \frac{1}{2} \sum |p_i|^2$. Following Konno, we define

$$\mathcal{S} = \left\{ S \subseteq \{1, \ldots, n\} \mid S \text{ is short} \right\}$$

and

$$\mathcal{S}' = \left\{ S \in \mathcal{S} \mid |S| \geq 2 \right\}.$$

Theorem 2.1 [K2] *The fixed point set* $X^{\mathbb{C}^*} = X^{S^1} = M \cup \bigcup_{S \in \mathcal{S}'} X_S$, *where*

$$X_S = \left\{ [p, q] \mid S \text{ and } S^c \text{ are each straight, and } p_j = 0 \text{ for all } j \in S^c \right\}.$$

Furthermore, X_S is diffeomorphic to $\mathbb{C}P^{|S|-2}$.

For all $S \in \mathcal{S}'$, let U_S be the closure inside of X of the set

$$\left\{ [p, q] \mid \lim_{\lambda \to \infty} \lambda \cdot [p, q] \in X_S \right\},$$

and let

$$\mathfrak{L} = M \cup \bigcup_{S \in \mathcal{S}'} U_S.$$

This reducible subvariety is called the *core* of X. Since $\lim_{\lambda \to 0} \lambda \cdot [p, q]$ always exists, the core is simply the union of those Białynicki-Birula strata whose closures are compact.[3] The $\mathbb{C}^*$ action defines an S^1-equivariant deformation retraction of X onto $\mathfrak{L}$ [N1].

---

[3]In Morse theoretic language, U_S is the closed flow-down set for X_S with respect to the Morse-Bott function Φ, and C is the union of all of the flow-down sets.

Theorem 2.2 *The core component U_S is smooth of complex dimension $n - 3$, and we have*

$$U_S = \big\{[p, q] \ \big| \ S \text{ is straight, and } p_j = 0 \text{ for all } j \in S^c\big\}.$$

Before proving Theorem 2.2, we describe the way in which the various components of the core fit together. For all $S \in \mathcal{S}'$, let

$$M_S = U_S \cap M = \big\{[0, q] \ \big| \ S \text{ is straight}\big\}.$$

We call this space the *polygon subspace* of M corresponding to the short subset S. Note that M_S is itself a polygon space with $n - |S| + 1$ edges, of lengths $\{\alpha_j \mid j \in S^c\} \cup \{\sum_S \alpha_i\}$. In particular, it is smooth. Now suppose given two short subsets $S, T \in \mathcal{S}'$, and consider the intersection $U_S \cap U_T$.

- If $S \cap T = \emptyset$, then $U_S \cap U_T = M_S \cap M_T$, a polygon subspace both of M_S and of M_T.

- If $S \cap T \neq \emptyset$ and $S \cup T$ is long, then $U_S \cap U_T = \emptyset$.

- If $S \cap T \neq \emptyset$ and $S \cup T$ is short, then

$$U_S \cap U_T = \big\{[p, q] \ \big| \ S \cup T \text{ is straight, and } p_j = 0 \text{ for all } j \in (S \cap T)^c\big\}.$$

  This is a subvariety of $U_{S \cup T}$ given by taking the closure inside of $U_{S \cup T}$ of a certain subbundle of the conormal bundle to $M_{S \cup T} \subseteq M$, defined by setting $p_j = 0$ for all $j \in (S \cap T)^c \supseteq (S \cup T)^c$.

Each of these descriptions generalizes to higher intersections without modification.

Finally, we compute the fixed point set $U_S^{\mathbb{C}^*}$. If $[p, q] \in U_S^{\mathbb{C}^*}$, then either $p = 0$ and $[p, q] \in M_S$, or $[p, q] \in X_T$ for some $T \in \mathcal{S}'$. If $[p, q] \in X_T$ then Theorem 2.1 tells us that T and T^c are each straight, hence $S \subseteq T$ or $S \subseteq T^c$. Since $p \neq 0$, we must have $S \subseteq T$. Indeed, $U_S \cap X_T$ is the linear subspace of $X_T \cong \mathbb{C}P^{|T|-2}$ given by the condition $p_j = 0$ for all $j \in T \setminus S$. In particular, $U_S \cap X_T$ is isomorphic to $\mathbb{C}P^{|S|-2}$ for any $T \supseteq S$.

Example 2.3 Let $n = 5$, $\alpha_1 = \alpha_2 = 1$, and $\alpha_3 = \alpha_4 = \alpha_5 = 3$, and consider the short subset $S = \{1, 2\}$. The fixed point set of U_S consists of $M_S \cong \mathbb{C}P^1$, and four points X_S, $U_S \cap X_{T_3}$, $U_S \cap X_{T_4}$, and $U_S \cap X_{T_5}$, where $T_j = \{1, 2, j\}$ for $j = 3, 4, 5$. For each j, $U_S \cap U_{T_j}$ is isomorphic to $\mathbb{C}P^1$, and touches M_S at the point M_{T_j}. In the following picture, an ellipse represents a copy of $\mathbb{C}P^1$ flowing between two fixed points, where the numbers or pairs of numbers indicate subsets that are straight on this $\mathbb{C}P^1$. (For example, $12, 45$ means that 1 and 2 are straight, as are 4 and 5.) We will revisit this example at the end of Section 4.

Proof of 2.2: The fact that $\dim U_S = \frac{1}{2} \dim X = n - 3$ is a general property of core components of quiver varieties [N1]. Thus, by a dimension count, it is enough to show that the set $\big\{[p, q] \ \big| \ S \text{ is straight, and } p_j = 0 \text{ for all } j \in S^c\big\}$ is contained in U_S.
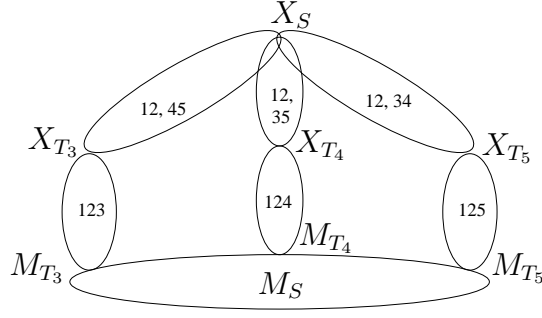
Figure 4: U_S, with $S = \{1, 2\}$

Consider a point $[p, q] \in X$ with S straight, and $p_j = 0$ for all $j \in S^c$. By applying an element of G, we may assume that $q_i = \binom{1}{0}$ for all $i \in S$. Suppose further that there exists an $i \in S$ with $p_i \neq 0$, and that no strict superset of S is straight. In other words, if $q_j = \binom{a_j}{b_j}$ for $j \in S^c$, suppose that $b_j \neq 0$. For $\lambda \in \mathbb{C}^*$, let $A(\lambda) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix} \right)$, let $e_i(\lambda) = \lambda$ for all $i \in S$, and let $e_j(\lambda) = \lambda^{-1}$ for all $j \in S^c$. Then for $i \in S$, we have $e_i(\lambda)^{-1} p_i A(\lambda) = \lambda^{-2} p_i$ and $A(\lambda)^{-1} q_i e_i = q_i$. For $j \in S^c$, we have $A(\lambda)^{-1} q_j e_j = \binom{\lambda^{-2} a_j}{b_j}$. Hence

$$
\begin{aligned}
\lim_{\lambda \to \infty} \lambda \cdot [p, q] &= \lim_{\lambda \to \infty} \lambda^2 \cdot [p, q] \\
&= \lim_{\lambda \to \infty} [\lambda^2 p, q] \\
&= \lim_{\lambda \to \infty} [\lambda^2 e(\lambda)^{-1} p A(\lambda), A(\lambda)^{-1} q e(\lambda)^{-1}] \\
&= [p, q'],
\end{aligned}
$$

where $q_i' = q_i$ for $i \in S$, and $q_j' = \binom{0}{b_j}$ for $j \in S^c$. Since we have assumed that $b_j \neq 0$ for all $j \in S^c$ and that $p_i \neq 0$ for some $i \in S$, (p, q') is stable, and hence defines an element of X_S. Since U_S is defined to be the closure of the set of elements that flow up to X_S, it includes all $[p, q]$ with S straight and $p_j = 0$ for all $j \in S^c$.

To see that U_S is smooth, it is sufficient to show that U_S is smooth at $[p, q]$ for all $[p, q] \in X^{\mathbb{C}^*}$. First suppose that $[p, q] \in X_T$ for some $T \in \mathcal{S}'$ containing S. Suppose, without loss of generality, that $T = \{1, \ldots, l\}$ and $S = \{1, \ldots, m\}$ for some $l \leq m$. Konno computes an explicit local complex chart for X at the point $[p, q]$, with coordinates $\{z_i, w_i \mid 3 \leq i \leq n-1\}$ [K2]. With respect to these coordinates, a point $[p', q']$ has the property that S is straight and $p_j' = 0$ for all $j \in S^c$ if and only if $w_i = 0$ for all $3 \leq i \leq l$ and $z_j = 0$ for all $l+1 \leq j \leq n-1$. Hence U_S is smooth at $[p, q]$.

It remains only to show that U_S is smooth at $M_S = U_S \cap M$. Let

$$
E = \{(p, q) \mid S \text{ is straight}, \ p_j = 0 \text{ for all } j \in S^c, \text{ and } \mu_{\mathbb{C}}(p, q) = 0\},
$$

9

and let $N = \{(p, q) \in E \mid p = 0\}$. The natural projection from E to N exhibits E as a vector bundle over N, because the equation $\mu_{\mathbb{C}}(p, q) = 0$ is linear in p. By definition, $U_S = E /\!\!/ G = E^{\alpha-\text{st}}/G$, and $M_S = N /\!\!/ G = N^{\alpha-\text{st}}/G$. The set $E|_{N^{\alpha-\text{st}}}/G \subseteq E^{\alpha-\text{st}}/G$ is an open neighborhood of M_S inside of U_S, and is isomorphic to a vector bundle over M_S. Since M_S is a polygon space it is smooth, hence U_S is smooth in a neighborhood of M_S. $\quad\square$

The following corollary is known to the experts; we include it here for lack of an explicit reference.

Corollary 2.4 U_S *is a compactification of the conormal bundle to M_S in M.*

Proof: We must show that the normal bundle to M_S in U_S is dual to the normal bundle to M_S in M. We use only general facts about quiver varieties from [N1], and the additional information that U_S is smooth, from Theorem 2.2. Consider a point $[0, q] \in M_S$, and let H_0 and H_1 be the 0 and 1 weight spaces of the $\mathbb{C}^*$ action on $T_{[0,q]}X$. The holomorphic symplectic form $\omega_{\mathbb{C}}$ is being rotated by $\mathbb{C}^*$ with weight 1 [N1, 5.1], hence it defines a perfect pairing between H_0 and H_1. The fiber at $[0, q]$ of the normal bundle to M_S in M is $H_0/T_{[0,q]}M_S$, which is dual by $\omega_{\mathbb{C}}$ to the annihilator of $T_{[0,q]}M_S$ inside of H_1. Since U_S is $\mathbb{C}^*$-invariant, we may write

$$T_{[0,q]}U_S = T_{[0,q]}U_S \cap H_0 \oplus T_{[0,q]}U_S \cap H_1 = T_{[0,q]}M_S \oplus T_{[0,q]}U_S \cap H_1.$$

To prove Corollary 2.4, we must show that $T_{[0,q]}U_S \cap H_1$ is equal to the annihilator of $T_{[0,q]}M_S$. The fact that $T_{[0,q]}U_S \cap H_1$ is contained in the annihilator of $T_{[0,q]}M_S$ follows from the fact that U_S is lagrangian with respect to $\omega_{\mathbb{C}}$ [N1] (here we use smoothness of U_S at $[0, q]$). Equality is then obtained by dimension count. $\quad\square$

We next describe U_S in polygon-theoretic terms, as a certain moduli space of pairs of polygons in $\mathbb{R}^3$.

Theorem 2.5 *Let U_S be the component of the core of X corresponding to a subset $S \in \mathcal{S}'$. Then U_S is homeomorphic to the moduli space of $n + 1$ vectors*

$$\{u_i, v_j, w \in \mathbb{R}^3 \mid i \in S, j \in S^c\},$$

taken up to rotation, satisfying the following conditions:

$$1) \quad w + \sum_{j \in S^c} v_j = 0$$

$$2) \quad \sum_{i \in S} u_i = 0$$

$$3) \quad u_i \cdot w = 0 \quad \text{for all } i \in S$$

$$4) \quad \|v_j\| = \alpha_j \quad \text{for all } j \in S^c$$

$$5) \quad \|w\| = \sum_{i \in S} \sqrt{\alpha_i^2 + \|u_i\|^2}.$$

Remark 2.6 In more descriptive terms, a point in U_S specifies two polygons in $\mathbb{R}^3$, as in Figure 5. The first is the $n - |S| + 1$ sided polygon consisting of the vectors $\{v_j \mid j \in S^c\}$ and
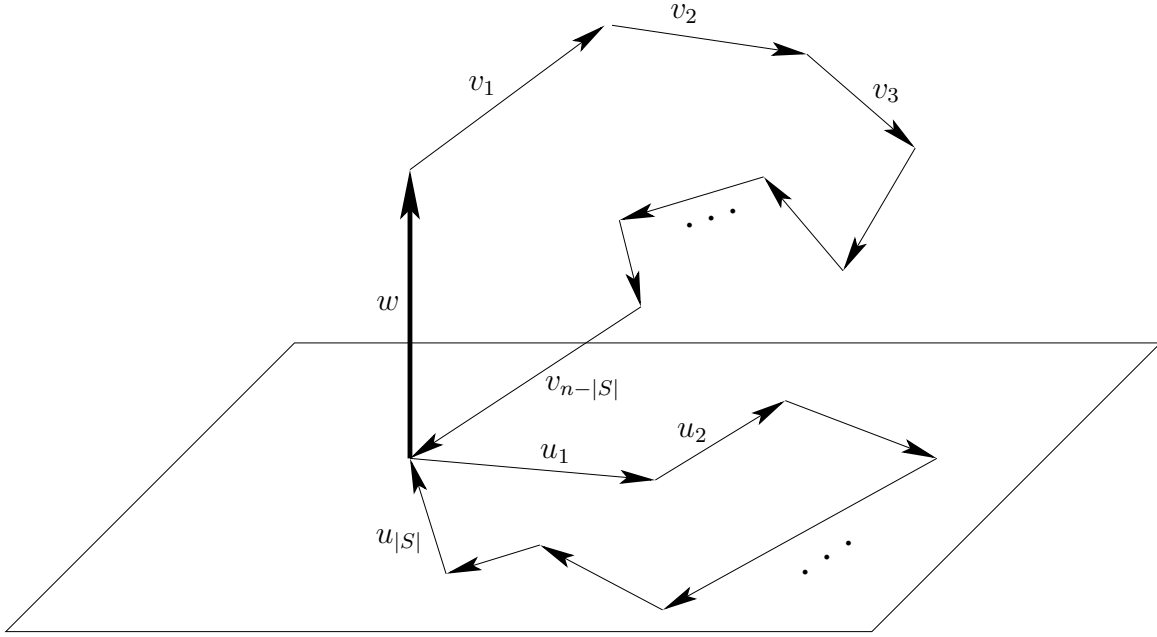

Figure 5:   An element of U_S, represented by a spatial polygon with a distinguished edge, and a planar polygon perpendicular to that edge.

w. Each vector v_j has length α_j, and w has a variable length, always greater than or equal to $\sum_{i \in S} \alpha_i$. This variable length is determined by the lengths of the edges in the second polygon, which consists of $|S|$ vectors $\{u_i \mid i \in S\}$, all contained in the plane perpendicular to w. Note that this description also applies to the Kähler polygon space M by taking $S = \emptyset$.

By setting $u_i = 0$ for all i we get M_S, the minimum of the Morse-Bott function Φ on U_S. On the other hand, consider the submanifold of U_S obtained by imposing the extra condition that $\|w\| = \sum_{j \in S^c} \|v_j\|$. Then the first of the two polygons is forced to be linear, and we are

11

left with $|S|$ vectors $\{u_i\}$ in the perpendicular plane satisfying a certain norm condition and adding to zero. Identifying this plane with $\mathbb{C}$ and dividing by the circle action rotating this plane, we obtain $\mathbb{C}P^{|S|-2} \cong X_S$, the maximum of Φ on U_S. Other critical points of Φ occur whenever the first polygon is linear, which is possible for finitely many values of $\|w\|$.

Proof of 2.5: Suppose given a point $[p, q]_\mathbb{R} \in U_S$, and let

$$u_i = q_i p_i + p_i^* q_i^* \quad \text{for all } i \in S,$$

$$v_j = (q_j q_j^*)_0 \quad \text{for all } j \in S^c,$$

$$w = \sum_{i \in S} (q_i q_i^*)_0 - (p_i^* p_i)_0.$$

These vectors live in $i \cdot \mathfrak{su}(2) \cong \mathfrak{su}(2)^* \cong \mathbb{R}^3$, which is endowed with the metric $A \cdot B = \frac{1}{2} \operatorname{tr} AB$, invariant under the coadjoint action. With respect to this metric, we have the equalities $\|(qq^*)_0\| = \frac{1}{2}|q|^2$ and $\|(p^*p)_0\| = \frac{1}{2}|p|^2$, hence conditions (1), (2), and (4) are immediate consequences of the moment map equations.

To verify condition (3), note that the vectors $\{q_i \mid i \in S\}$ are all proportional over $\mathbb{C}$, which implies that the vectors $(q_i q_i^*)_0$ are positive scalar multiples of each other. Furthermore, the moment map equation $p_i q_i = 0$ implies that $(p_i^* p_i)_0$ is a non-positive scalar multiple of $(q_i q_i^*)_0$, therefore $w = \sum (q_i q_i^*)_0 - (p_i^* p_i)_0$ is proportional over $\mathbb{R}_+$ to $(q_i q_i^*)_0$ for any $i \in S$. Then $u_i \cdot w = \frac{1}{2} \operatorname{tr} u_i w$ is a multiple of

$$\operatorname{tr} u_i(q_i q_i^*)_0 = \operatorname{tr} u_i q_i q_i^* = \operatorname{tr} p_i^* q_i^* q_i q_i^* = |q_i|^2 \operatorname{tr} p_i^* q_i^* = 0,$$

where the first equality comes from the fact that $q_i q_i^* - (q_i q_i^*)_0$ is a scalar multiple of the identity, and $\operatorname{tr} u_i = 0$.

To check condition (5), we first compute the norm of u_i:

$$
\begin{aligned}
\|u_i\|^2 &= \frac{1}{2} \operatorname{tr} u_i^2 \\
&= \frac{1}{2} \operatorname{tr}(q_i p_i p_i^* q_i^* + p_i^* q_i^* q_i p_i) \\
&= |q_i|^2 |p_i|^2 \\
&= |q_i|^2 (|q_i|^2 - 2\alpha_i).
\end{aligned}
$$

Since all of the vectors $\{(q_i q_i^*)_0 - (p_i^* p_i)_0 \mid i \in S\}$ point in the same direction, we have

$$\|w\| = \sum_{i \in S} \|(q_i q_i^*)_0\| + \|(p_i^* p_i)_0\| = \sum_{i \in S} \frac{1}{2}|q_i|^2 + \frac{1}{2}|p_i|^2 = \sum_{i \in S} |q_i|^2 - \alpha_i = \sum_{i \in S} \sqrt{\alpha_i^2 + \|u_i\|^2}.$$

We have defined a map from U_S to the moduli space of vectors $\{u_i, v_j, w\}$ satisfying conditions (1)-(5), and we claim that this map is a homeomorphism. Since the source of this

map is compact and the target is Hausdorff, it is sufficient to show that the map is bijective.

Suppose given a collection of vectors $\{u_i, v_j, w\} \subseteq \mathfrak{su}(2)$ satisfying conditions (1)-(5). Using the adjoint action of $SU(2)$, we may assume that w is a positive scalar multiple of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By condition (3), this implies that for all $i \in S$, there exists $\lambda_i \in \mathbb{C}$ with $u_i = \begin{pmatrix} 0 & \lambda_i \\ \bar{\lambda}_i & 0 \end{pmatrix}$. For $j \in S^c$, we choose $q_j \in \mathbb{C}^2$ with $(q_j q_j^*)_0 = v_j$, and observe that q_j is unique up to the action of $U(1)^n$. We know that for all $i \in S$, $(q_i q_i^*)_0$ must be a positive multiple of w, hence there exist $a_i, b_i \in \mathbb{C}$ such that

$$q_i = \begin{pmatrix} a_i \\ 0 \end{pmatrix} \quad \text{and} \quad p_i = (0 \ b_i)$$

for all $i \in S$. In order to have $u_i = q_i p_i + p_i^* q_i^*$ and $\frac{1}{2}|q_i|^2 - \frac{1}{2}|p_i|^2 = \alpha_i$, we must have

$$a_i b_i = \lambda_i \quad \text{and} \quad \frac{1}{2}|a_i|^2 - \frac{1}{2}|b_i|^2 = \alpha_i.$$

These equations uniquely define a_i and b_i up to the action of $U(1)^n$. It follows from conditions (1)-(5) that $(p, q) \in \mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)$ and that $w = \sum_{i \in S}(q_i q_i^*)_0 - (p_i^* p_i)_0$. This shows that our map is bijective, and thus completes the proof of Theorem 2.5. $\qquad\square$

Remark 2.7 Suppose that S has only two elements; without loss of generality we will assume that $S = \{1, 2\}$. Then forgetting u_1 and u_2 gives a diffeomorphism from U_S to the "vertical polygon space" $VP(\alpha_3, \ldots, \alpha_n, \alpha_1 + \alpha_2)$ defined in [HK2], shown to be diffeomorphic to a toric variety. More generally with $S = \{1, \ldots, k\}$, given any two-element subset $T \subseteq S$, the subvariety of U_S given by the equations $u_i = 0$ for all $i \in S \setminus T$ is diffeomorphic to $VP(\alpha_{k+1}, \ldots, \alpha_n, \sum_T \alpha_i)$.

3 Equivariant cohomology of X

We begin by defining representations

$$\rho_{ij} : K \to U(1) \quad \text{and} \quad \rho_{SO(3)} : K \to SO(\mathfrak{su}(2))$$

by the formulae

$$\rho_{ij}[A; e_1, \ldots, e_n] = e_i e_j \quad \text{and} \quad \rho_{SO(3)}[A; e_1, \ldots, e_n] = \mathrm{Ad}(A).$$

Associated to these representations are the vector bundles

$$\mathcal{L}_{ij} = \left(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\right) \times_{\rho_{ij}} \mathbb{C} \quad \text{and} \quad \mathcal{E} = \left(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\right) \times_{\rho_{SO(3)}} \mathfrak{su}(2).$$

Let $c_i = c_1(\mathcal{L}_{ii}) \in H^2(X)$ and $p = p_1(\mathcal{E}) \in H^4(X)$.

Theorem 3.1 [K2] *The cohomology ring $H^*(X)$ is isomorphic to $\mathbb{Q}[c_1, \ldots, c_n, p]/\mathcal{I}$, where $\mathcal{I}$ is generated by the following two families:*

$$\begin{aligned} &1) \quad p - c_i^2 \quad \textit{for all} \quad i \in \{1, \ldots, n\} \\ &2) \quad \textit{all elements of degree } 2(n-2). \end{aligned}$$

The action of S^1 on the total spaces of $\mathcal{L}_{ij}$ and $\mathcal{E}$ given by the left action on $\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)$ gives these bundles an S^1-equivariant structure. For the rest of this section, we will use c_i and p to denote the *equivariant* characteristic classes

$$c_i = c_1(\mathcal{L}_{ii}) \in H_{S^1}^2(X) \quad \text{and} \quad p = p_1(\mathcal{E}) \in H_{S^1}^4(X).$$

Let $\mathcal{K}$ be the S^1-equivariant line bundle on X obtained by pulling back the weight 1 line bundle over a point, and let

$$x = c_1(\mathcal{K}) \in H_{S^1}^2(X).$$

We obtain the following result, extending Konno's work to the equivariant context.

Theorem 3.2 *The equivariant cohomology ring $H_{S^1}^*(X)$ is isomorphic to $\mathbb{Q}[c_1, \ldots, c_n, p, x]/\mathcal{J}$, where $\mathcal{J}$ is generated by the following two families:*

$$\begin{aligned} &1) \quad p - c_i^2 \quad \textit{for all} \quad i \in \{1, \ldots, n\} \\ &2) \quad \prod_{j \in \overline{S^c}} (c_j + c_{n_S}) \times \prod_{i \in \overline{S}} (c_i + x) \quad \textit{for all} \quad \emptyset \neq S \in \mathcal{S}, \end{aligned}$$

where $m_S \in S$ and $n_S \in S^c$ are the minimal elements of the two sets, $\overline{S} = S \smallsetminus \{m_S\}$, and $\overline{S^c} = S^c \smallsetminus \{n_S\}$.

Remark 3.3 Konno observes that the quotient map from the abstract polynomial ring $\mathbb{Q}[c_1, \ldots, c_n, p]$ to $H^*(X)$ is precisely the Kirwan map

$$\kappa : H_K^*(T^*\mathbb{C}^{2n}) \to H^*(X)$$

induced by the inclusion $\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0) \hookrightarrow T^*\mathbb{C}^{2n}$. Theorem 3.1 can be interpreted as saying that the Kirwan map for hyperpolygon spaces is surjective, with kernel $\mathcal{I}$. Likewise, Theorem 3.2 asserts that the S^1-equivariant Kirwan map

$$\kappa_{S^1} : H_{K \times S^1}^*(T^*\mathbb{C}^{2n}) \to H_{S^1}^*(X)$$

is surjective, with kernel $\mathcal{J}$. The analogous map for Kähler reductions is known to always be surjective [Ki], but in the hyperkähler case the problem remains open.

Remark 3.4 The second family of relations in Theorem 3.2 has a geometric interpretation. In the course of the proof of Theorem 3.2 it will be shown that the class $-\frac{1}{2}(c_j + c_{n_s})$ is represented by the divisor given by the points $[p, q] \in X$ where q_j and q_{n_s} are straight. Hence the product $\prod_{i \in \overline{S^c}}(c_j + c_{n_s})$ is supported on the subvariety of points where S^c is straight. Similarly, it is shown in the proof that the class $c_i + x$ is represented by the divisor given by the condition $p_i = 0$, hence the class

$$\prod_{j \in \overline{S^c}}(c_j + c_{n_s}) \times \prod_{i \in \overline{S}}(c_i + x)$$

is supported on the subvariety of points $[p, q]$ where a long subset S^c is straight, and *all but one p_i for $i \in S$ is zero*. We will show, using Theorem 2.1, that this subvariety is disjoint from the fixed point set X^{S^1}, and that this implies that its cohomology class is trivial.

Before proceeding with the proof of Theorem 3.2, we collect some preliminary results regarding the relationship between ordinary and equivariant cohomology. The first result that we need is known as *equivariant formality*, proven for compact manifolds in [Ki], which we adapt to our situation in Proposition 3.5.

Proposition 3.5 *Let X be a symplectic manifold, possibly noncompact but of finite topological type. Suppose that X admits a hamiltonian circle action, and that the moment map is proper and bounded below. Then $H^*_{S^1}(X)$ is a free module over $H^*_{S^1}(pt)$.*

Proof: Because Φ is a moment map, it is a Morse-Bott function such that all of the critical submanifolds and their normal bundles carry almost complex structures. Thus we get a Morse-Bott decomposition of X into even-dimensional S^1-invariant submanifolds. This tells us, as in [Ki], that the spectral sequence associated to the fibration $X \hookrightarrow EG \times_G X \to BG$ collapses, and we get the desired result. $\qquad\square$

Consider the following commuting square of maps, where ϕ and ψ are each given by setting x to zero.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{J} & \longrightarrow & H^*_{K \times S^1}(T^*\mathbb{C}^{2n}) & \longrightarrow & H^*_{S^1}(X) \\
& & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} \\
0 & \longrightarrow & \mathcal{I} & \longrightarrow & H^*_K(T^*\mathbb{C}^{2n}) & \longrightarrow & H^*(X)
\end{array}
$$

Our moment map $\Phi : X \to \mathbb{R}$ is proper and bounded below [HP, 1.3], therefore Proposition 3.5 has the following consequences.

Corollary 3.6 *The equivariant Kirwan map κ_{S^1} is surjective.*

Proof: Suppose that $\gamma \in H^*_{S^1}(X)$ is a homogeneous class of minimal degree that is *not* in the image of κ_{S^1}, and choose a class $\eta \in \phi^{-1}\kappa^{-1}\psi(\gamma)$. Then by Proposition 3.5, $\kappa_{S^1}(\eta) - \gamma = x\delta$ for some $\delta \in H^*_{S^1}(M)$, and therefore δ is a class of lower degree that is not in the image of κ_{S^1}. $\qquad\square$

Corollary 3.7 *Suppose that* $\mathcal{J} \subseteq \ker \kappa_{S^1}$ *and* $\phi(\mathcal{J}) = \mathcal{I}$. *Then* $\mathcal{J} = \ker \kappa_{S^1}$.

Proof: Suppose that $a \in \ker \kappa_{S^1} \smallsetminus \mathcal{J}$ is a homogeneous class of minimal degree, and choose $b \in \mathcal{J}$ such that $\phi(a-b) = 0$. Then $a - b = cx$ for some $c \in H^*_{K \times S^1}(T^*\mathbb{C}^{2n})$. By Proposition 3.5, $cx \in \ker \kappa_{S^1} \Rightarrow c \in \ker \kappa_{S^1}$, hence $c \in \ker \kappa_{S^1} \smallsetminus \mathcal{J}$ is a class of lower degree than a. $\square$

Corollary 3.8 *Let E be an S^1-equivariant vector bundle on X, with an equivariant section s such that the zero set of s is disjoint from the fixed point set X^{S^1}. Then $e(E) = 0 \in H^*_{S^1}(X)$.*

Proof: Consider the bundle $\mathcal{K}^{\oplus 2(n-3)}$ restricted to $X \smallsetminus X^{S^1}$. An equivariant section of this bundle is equivalent to an ordinary section of the induced bundle over the quotient $\left(X \smallsetminus X^{S^1}\right)/S^1$, and by degree considerations we can find a nonvanishing section. Hence $\mathcal{K}^{\oplus 2(n-3)}$ has an equivariant section t over X with zero set supported on X^{S^1}. Then $s \oplus t$ is a nonvanishing section of $E \oplus \mathcal{K}^{\oplus 2(n-3)}$, hence

$$0 = e\left(E \oplus \mathcal{K}^{\oplus 2(n-3)}\right) = e(E) \cdot e\left(\mathcal{K}^{\oplus 2(n-3)}\right) = x^{2(n-3)}e(E).$$

Then by Proposition 3.5, $e(E) = 0$. $\qquad\square$

Remark 3.9 Corollary 3.8 is a weak form of the statement that the restriction map from $H^*_{S^1}(X)$ to $H^*_{S^1}(X^{S^1})$ is injective, proven for compact X in [Ki].

Proof of 3.2: Corollary 3.6 tells us that the characteristic classes $c_1, \ldots, c_n, p, x \in H^*_{S^1}(X)$ generate the ring, and Corollary 3.7 tells us that it is enough to prove two statements: the first is that $\mathcal{J} \subseteq \ker \kappa_{S^1}$, i.e. that the elements of $\mathcal{J}$ are indeed relations in $H^*_{S^1}(X)$, and the second is that $\phi(\mathcal{J}) = \mathcal{I}$. We begin by proving that $\mathcal{J} \subseteq \ker \kappa_{S^1}$, following the approach outlined in Remark 3.4.

In the nonequivariant context, Konno shows that $\mathcal{E} \cong \mathcal{L}_{ii} \oplus \mathbb{R}$ as a real vector bundle for all i [K2]. This implies that

$$p = p_1(\mathcal{E}) = -c_2(\mathcal{E} \otimes \mathbb{C}) = -c_2(\mathcal{L}_{ii} \oplus \mathcal{L}^*_{ii}) = c_1(\mathcal{L}_{ii})^2 = c_i^2 \in H^*(X),$$

and therefore $p - c_i^2 \in \ker \kappa$. This argument is adaptable to the equivariant context without any modifications, hence we will not include it here.

Consider the function

$$\tilde{s}_{ij} : \mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0) \to \mathbb{C}$$

16

given by $\tilde{s}_{ij}(p,q) = \det(q_iq_j)$, where (q_iq_j) is considered to be a 2×2 matrix [K2]. This function is S^1-invariant, and K-equivariant with respect to ρ_{ij}, and therefore defines an S^1-equivariant section s_{ij} of $\mathcal{L}_{ij}^*$ (the dualization is a consequence of the fact that the action of K on $\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)$ is a right action). The vanishing set of s_{ij} is the divisor

$$Z_{ij} = \big\{ [p,q] \in X \ \big| \ q_i \text{ is proportional to } q_j \big\},$$

hence $c_1(\mathcal{L}_{ij}^*) = -\frac{1}{2}(c_i + c_j)$ is represented in equivariant Borel-Moore homology by the divisor Z_{ij}. Now consider the function

$$\tilde{t}_i : \mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0) \to \mathbb{C}$$

given by

$$\tilde{t}_i(p,q) = \begin{cases} p_i^{(2)}/q_i^{(1)} & \text{if } q_i^{(1)} \neq 0 \\ -p_i^{(1)}/q_i^{(2)} & \text{if } q_i^{(2)} \neq 0, \end{cases}$$

where

$$q_i = \begin{pmatrix} q_i^{(1)} \\ q_i^{(2)} \end{pmatrix} \quad \text{and} \quad p_i = (p_i^{(1)} p_i^{(2)}).$$

This function is well-defined by the fact that $\mu_{\mathbb{C}}(p,q) = 0$. It is S^1-equivariant with respect to the weight 1 action of S^1 on $\mathbb{C}$, and it is K-equivariant with respect to $\bar{\rho}_{ii}$. Thus it descends to an equivariant section t_i of $\mathcal{L}_{ii} \otimes \mathcal{K}$, vanishing on the divisor

$$W_i = \big\{ [p,q] \in X \ \big| \ p_i = 0 \big\},$$

so $c_i + x = c_1(\mathcal{L}_{ii} \otimes \mathcal{K})$ is represented by the divisor W_i.

Consider a nonempty short subset $S \in \mathcal{S}$, and define the vector bundle

$$E_S = \bigoplus_{j \in \overline{S^c}} \mathcal{L}_{jn_S}^* \ \oplus \ \bigoplus_{i \in \overline{S}} \mathcal{L}_{ii} \otimes \mathcal{K}$$

with equivariant Euler class

$$(-1/2)^{|\overline{S^c}|} \prod_{j \in \overline{S^c}} (c_j + c_{n_S}) \times \prod_{i \in \overline{S}} (c_i + x).$$

Then $(\oplus_{i \in \overline{S}} t_i) \oplus_{j \in \overline{S^c}} s_{jn_S}$ is a section of E_S vanishing only on the cycle

$$Z_S = \bigcap_{i \in \overline{S}} Z_i \ \cap \ \bigcap_{j \in \overline{S^c}} Z_{jn_S}$$

consisting of points $[p,q]$ such that q_j is proportional to q_{n_S} for all $j \in S^c$, and $p_i = 0$ for all $i \in \overline{S}$. We would like to show that $e(E_S) = 0 \in H_{S^1}^*(X)$, and by Corollary 3.8, it will suffice

to show that Z_S is disjoint from

$$X^{S^1} = M \cup \bigcup_{T \in \mathcal{S}} X_T.$$

Since S^c is a long subset that is straight in Z_S, we have $Z_S \cap M = \emptyset$ by Theorem 1.1. We must now show that Z_S also does not intersect any $X_T \subset X^{S^1}$. We begin with the observation that for each $X_T, T \in \mathcal{S}'$,

$$[p, q] \in X_T \Rightarrow \text{ at least } two \text{ of the vectors in } \{p_i \mid i \in T\} \text{ are } nonzero. \tag{2}$$

This follows from the description of X_T given in Theorem 2.1 and the complex moment map conditions. Now let $T \in \mathcal{S}'$ be a short subset. If $T \nsubseteq S$, then by the descriptions of X_T and Z_S, we may conclude that any point $[p, q] \in Z_S \cap X_T$ must have the long subset $T \cup S^c$ straight, and $p_j = 0$ for all $j \in T^c$. This means that (p, q) is unstable by Theorem 1.1, so $Z_S \cap X_T = \emptyset$. On the other hand, if $T \subseteq S$, then we may similarly conclude that for any point $[p, q] \in Z_S \cap X_T$, the long subset T^c is straight, and $at \ most \ one$ element in $\{p_i \mid i \in T\}$ is nonzero. This contradicts the observation (2) above, so $Z_S \cap X_T = \emptyset$.

We have now proven that $\mathcal{J} \subseteq \ker \kappa_{S^1}$, and it remains only to show that $\phi(\mathcal{J}) = \mathcal{I}$. In other words (by Theorem 3.1), we must show that the set

$$\left\{ \prod_{j \in \overline{S^c}} (c_j + c_{n_S}) \times \prod_{i \in \overline{S}} c_i \ \middle| \ \emptyset \neq S \in \mathcal{S} \right\}$$

spans all monomials of degree $2(n-2)$ in the ring $\mathbb{Q}[c_1, \ldots, c_n] / \langle c_i^2 - c_1^2 \mid i \in \{2, \ldots, n\} \rangle$.

Let $b_k = \frac{1}{2}(c_1 + c_k)$ for all k, so that $c_k = 2b_k - b_1$. The relations $c_k^2 = c_1^2$ translate to $b_k^2 = b_1 b_k$ for all k. Let

$$v_S = \frac{(-1)^n}{2^{|\overline{S^c}|}} \prod_{j \in \overline{S^c}} (c_j + c_{n_S}) \times \prod_{i \in \overline{S}} c_i = (-1)^n \prod_{j \in \overline{S^c}} (b_j + b_{n_S} - b_1) \times \prod_{i \in \overline{S}} (2b_i - b_1),$$

and let

$$b_A = (-1)^{|A|} b_1^{n-2-|A|} \prod_{k \in A} b_k$$

for all $A \subsetneq \{2, \ldots, n\}$. Then $\{b_A\}$ is a basis for the $(n-2)^{\text{nd}}$ graded piece of the ring

$$\mathbb{Q}[b_1, \ldots, b_n] / \langle b_k^2 - b_1 b_k \mid k \in \{2, \ldots, n\} \rangle,$$

hence we need to show that each element b_A can be expressed as a linear combination of the elements $\{v_S \mid S \in \mathcal{S}\}$.

Claim 3.10 *We have the following expression for v_S in terms of the basis $\{b_A\}$:*

$$v_S = \begin{cases} \displaystyle\sum_{\substack{\overline{S^c} \subseteq A \\ m_S \notin A}} 2^{|A \cap \overline{S}|} \, b_A & \text{if } 1 \in S^c; \\[2em] \displaystyle\sum_{S^c \nsubseteq A} 2^{|A \cap \overline{S}|} \, b_A & \text{if } 1 \in S. \end{cases}$$

Proof: Any degree $n-2$ monomial in $b_1, \ldots, b_n$ is equal to $(-1)^{|A|} b_A$, where A is the set of $k > 1$ such that b_k appears in the monomial. Expanding v_S, we need to count (with sign) the occurrence of b_A for each A. In most cases we find that there is no cancellation, and the claim is straightforward. The most difficult case occurs when $1 \in S$ (therefore $n_S = 1$) and $n_S \in A$; in this case the number of times (with multiplicity) that b_A occurs in v_S is

$$(-1)^n (-1)^{|A|} (-1)^{|A^c \cap \overline{S}|} \, 2^{|A \cap \overline{S}|} \sum_{E \subsetneq A^c \cap \overline{S^c}} (-1)^{|E|}$$

$$= (-1)^n (-1)^{|A|} (-1)^{|A^c \cap \overline{S}|} \, 2^{|A \cap \overline{S}|} \left((1-1)^{|A^c \cap \overline{S^c}|} - (-1)^{|A^c \cap \overline{S^c}|} \right)$$

$$= (-1)^{n + |A| + |A^c \cap \overline{S}| + |A^c \cap \overline{S^c}| + 1} \, 2^{|A \cap \overline{S}|}$$

$$= (-1)^{2n} \, 2^{|A \cap \overline{S}|}$$

$$= 2^{|A \cap \overline{S}|}.$$

(When we write A^c, we mean the complement of A inside of $\{2, \ldots, n\}$.) We leave the remaining cases to be checked by the reader. $\qquad\square$

Claim 3.11 *Suppose that $1 \in S$. Let $S_0 = S$, and for $1 \le k \le |S|$, let $S_k = S_{k-1} \setminus \{m_{S_{k-1}}\}$ (i.e. S_k consists of the $|S|-k$ largest elements of S). Then $\displaystyle v_S + \sum_{k=1}^{|S|-1} 2^{k-1} v_{S_k} = \sum_A 2^{|A \cap \overline{S}|} \, b_A$.*

Proof: We proceed by induction to show that

$$v_S + \sum_{k=1}^{l} 2^{k-1} v_{S_k} = \sum_A 2^{|A \cap \overline{S}|} \, b_A - 2^l \sum_{\overline{S_{l+1}^c} \subseteq A} 2^{|A \cap \overline{S_l}|} \, b_A;$$

the case $l = |S| - 1$ is the statement of the claim. The base case $l = 0$ follows from Claim 3.10, together with the observation that $\overline{S_1^c} = S^c$. More generally, for all $l \ge 1$, we have

19

$\overline{S^c_{l+1}} = S^c \cup \{m_{S_1}, \ldots, m_{S_l}\}$. Then

$$v_S + \sum_{k=1}^{l+1} 2^{k-1} \, v_{S_k} \;=\; v_S + \sum_{k=1}^{l} 2^{k-1} \, v_{S_k} + 2^l \, v_{S_{l+1}}$$

$$= \sum_A 2^{|A \cap \overline{S}|} \, b_A - 2^l \sum_{\overline{S^c_{l+1}} \subseteq A} 2^{|A \cap \overline{S}_l|} \, b_A + 2^l \sum_{\substack{\overline{S^c_{l+1}} \subseteq A \\ m_{S_{l+1}} \notin A}} 2^{|A \cap \overline{S}_{l+1}|} \, b_A$$

by the inductive hypothesis and Claim 3.10. Using the fact that $A \cap \overline{S}_{l+1} = A \cap \overline{S}_l$ when $m_{S_{l+1}} \notin A$, this is equal to

$$\sum_A 2^{|A \cap \overline{S}|} \, b_A - 2^l \sum_{\overline{S^c_{l+1}} \cup \{m_{S_{l+1}}\} \subseteq A} 2^{|A \cap \overline{S}_l|}.$$

Finally, since $|A \cap \overline{S}_{l+1}| = |A \cap \overline{S}_l| - 1$ when $m_{S_{l+1}} \in A$, this reduces to

$$\sum_A 2^{|A \cap \overline{S}|} \, b_A - 2^{l+1} \sum_{\overline{S^c_{l+2}} \subseteq A} 2^{|A \cap \overline{S}_{l+1}|} \, b_A,$$

thus proving our claim. $\qquad\square$

For all short subsets T containing 1, let $w_T = \sum_A 2^{|A \cap \bar{T}|} \, b_A$, which by Claim 3.11 is expressible as a linear combination of elements of the set $\{v_S \mid \emptyset \neq S \in \mathcal{S}\}$. Let

$$x_S = \begin{cases} \displaystyle\sum_{1 \in T \subseteq S} (-1)^{|S|+|T|} w_T & \text{if } 1 \in S, \\[2mm] v_S & \text{if } 1 \in S^c. \end{cases}$$

Our last task will be to prove that the transition matrix Q taking the basis $\{b_A\}$ to the set $\{x_S\}$ is upper triangular with ones on the diagonal, and therefore invertible. In order to make sense of "the diagonal," we must first give an explicit bijection between the set of proper subsets of $\{2, \ldots, n\}$ and the set of nonempty short subsets of $\{1, \ldots, n\}$. We do this as follows: given $A \subsetneq \{2, \ldots, n\}$, let

$$S(A) = \begin{cases} A^c & \text{if } A^c \text{ is short}, \\ \{1, \ldots, n\} \setminus A^c = A \cup \{1\} & \text{if } A^c \text{ is long}. \end{cases}$$

The rows of Q will be indexed by A, and the sets will appear in lexicographic order within cardinality class. For example, when $n = 4$, the order of the rows will be $\emptyset$, $\{2\}$, $\{3\}$, $\{4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$. The columns will be indexed by S according to the bijection described above.

Claim 3.12 *The matrix Q is lower triangular with ones on the diagonal.*

Proof: First consider a column corresponding to a short subset S that does *not* contain 1. The entries in this column correspond to the coefficient of b_A in $x_S = v_S$. From Claim 3.10, we see that b_A appears in v_S only if $\overline{S^c} \subseteq A \subseteq \overline{S^c} \cup \overline{S}$, and if so it appears with a coefficient of $2^{|A \cap \overline{S}|}$. Since $1 \notin S$, we have $\overline{S^c} = S^c \setminus \{1\} = \{2, \ldots, n\} \setminus S$. The diagonal entry corresponds to the set $A = \{2, \ldots, n\} \setminus S = \overline{S^c}$, therefore in this row we get the number $2^{|A \cap \overline{S}|} = 2^{|\overline{S^c} \cap \overline{S}|} = 1$. Since the set A corresponding to a given row can never contain the set B corresponding to a lower row, the rows above the diagonal fail to satisfy the condition $\overline{S^c} \subseteq A$, and we get all zeros.

Now consider a column corresponding to a short subset S that *does* contain 1. In this case, the coefficient of b_A in x_S is

$$(-1)^{|S|} \sum_{1 \in T \subseteq S} (-1)^{|T|} 2^{|A \cap \overline{T}|}.$$

The diagonal entry corresponds to the set $A = \overline{S}$, and we get

$$\begin{aligned}
(-1)^{|S|} \sum_{1 \in T \subseteq S} (-1)^{|T|} 2^{|\overline{T}|} &= (-1)^{|\overline{S}|} \sum_{1 \in T \subseteq S} (-2)^{|\overline{T}|} \\
&= (-1)^{|S|} (1-2)^{|\overline{S}|} = 1.
\end{aligned}$$

Any row above the diagonal corresponds to a set A which does not contain $\overline{S}$. Choose an element $l \in \overline{S} \setminus A$. Then

$$\begin{aligned}
(-1)^{|S|} \sum_{1 \in T \subseteq S} (-1)^{|T|} 2^{|A \cap \overline{T}|} &= (-1)^{|S|} \sum_{l \in T} (-1)^{|T|} 2^{|A \cap \overline{T}|} + (-1)^{|S|} \sum_{l \notin T} (-1)^{|T|} 2^{|A \cap \overline{T}|} \\
&= (-1)^{|S|} \sum_{l \notin T} \left[(-1)^{|T|} 2^{|A \cap \overline{T}|} + (-1)^{|T \cup \{l\}|} 2^{|A \cap \overline{T}|} \right] \\
&= 0.
\end{aligned}$$

$\square$

Claim 3.12 tells us that each b_A can be expressed as a linear combination of elements of the form x_S, and therefore of elements of the form v_S. This lets us conclude that $\phi(\mathcal{J}) = \mathcal{I}$, and thereby completes the proof of Theorem 3.2. $\square$

4 The cohomology ring of a core component

In this section we compute the S^1-equivariant and ordinary cohomology rings of the core component U_S corresponding to a short subset $S \subseteq \{1, \ldots, n\}$. Since U_S is the closure of

a cell in an even-dimensional equivariant cellular decomposition of X, the restriction map $H^*_{S^1}(X) \to H^*_{S^1}(U_S)$ is surjective. In particular, $H^*_{S^1}(U_S)$ is generated by restrictions of the Kirwan classes $c_1, \ldots, c_n, x$. For our presentation, it will be convenient to assume that $1 \in S$, and to work with the classes $b_k = \frac{1}{2}(c_1 + c_k)$ introduced in Section 3. With respect to these generators, we obtain the following result.

Theorem 4.1 *The equivariant cohomology ring $H^*_{S^1}(U_S)$ is isomorphic to $\mathbb{Q}[b_1, \ldots, b_n, x]/\mathcal{J}_S$, where $\mathcal{J}_S$ is generated by the following four families:*

$$
\begin{aligned}
&1) \quad b_1 - b_i \quad \text{for all} \quad i \in S \\
&2) \quad b_j(b_1 - b_j) \quad \text{for all} \quad j \in S^c \\
&3) \quad \prod_{j \in R} b_j \quad \text{for all} \quad R \subseteq S^c \text{ such that } R \cup S \text{ is long} \\
&4) \quad (b_1 + x)^{|S|-1} \cdot \frac{1}{b_1}\left(\prod_{j \in L}(b_j - b_1) - \prod_{j \in L} b_j\right) \quad \text{for all long subsets } L \subseteq S^c.
\end{aligned}
$$

Corollary 4.2 *The ordinary cohomology ring $H^*(U_S)$ is isomorphic to $\mathbb{Q}[b_1, \ldots, b_n]/\mathcal{I}_S$, where $\mathcal{I}_S$ is generated by the following four families:*

$$
\begin{aligned}
&1) \quad b_1 - b_i \quad \text{for all} \quad i \in S \\
&2) \quad b_j(b_1 - b_j) \quad \text{for all} \quad j \in S^c \\
&3) \quad \prod_{j \in R} b_j \quad \text{for all} \quad R \subseteq S^c \text{ such that } R \cup S \text{ is long} \\
&4) \quad b_1^{|S|-2} \prod_{j \in L}(b_j - b_1) \quad \text{for all long subsets } L \subseteq S^c.
\end{aligned}
$$

Remark 4.3 Each of these relations has a geometric interpretation. For $i \in \{1, \ldots, n\}$, it is possible to construct a line bundle on X with equivariant Euler class $b_i - b_1$ which has a section supported on the locus where $q_1 q_1^*$ and $q_i q_i^* \in \mathbb{R}^3$ point in opposite directions. Since this locus is disjoint from U_S when $i \in S$, we have

$$1) \ b_i = b_1 \in H^*_{S^1}(U_S) \ \text{ for all } \ i \in S.$$

Similarly, we showed in the proof of Theorem 3.2 that $-b_j = -\frac{1}{2}(c_1 + c_j)$ is represented by the divisor Z_{1j} on which $q_1 q_1^*$ and $q_i q_i^* \in \mathbb{R}^3$ point in the same direction. Then by the previous reasoning, we obtain

$$2) \ b_j(b_1 - b_j) = 0 \in H^*_{S^1}(U_S) \ \text{ for all } \ j \in S^c.$$

Recall from Section 3 that for any $R \subseteq S^c$, the cohomology class $(-1)^{|R|} \prod_{j \in R} b_j$ is represented by the subvariety $Z_R \subseteq X$ of points with q_j proportional to q_1 for all $j \in R$.

22

When restricted to U_S, this becomes $U_S \cap U_{R \cup S}$, the unstable manifold for the critical locus $X_{R \cup S} \cap U_S$ of the Morse-Bott function $\Phi|_{U_S}$. In particular, we have

$$3) \quad \prod_{j \in R} b_j = 0 \in H^*_{S^1}(U_S) \text{ if } R \cup S \text{ is long.}$$

To understand the fourth family of relations, recall from Section 3 that

$$b_1 + x = 2b_i - b_1 + x = c_i + x \in H^*_{S^1}(U_S)$$

is represented by the divisor W_i of points with $p_i = 0$ for any $i \in S$. In particular, the class $(b_1 + x)^{|S|-1}$ is represented by the subvariety of points in U_S on which $p_i = 0$ for all $i \in \overline{S}$, which is equal to M_S by the complex moment map condition. Hence the fourth family of generators of $\mathcal{J}_S$ (or of $\mathcal{I}_S$) can be interpreted geometrically as $(b_1 + x)^{|S|-1}$ (respectively $b_1^{|S|-1}$ in the nonequivariant case) times classes that vanish in $H^*_{S^1}(M_S)$ (see Lemma 4.5).

Proof of 4.1: Let $\phi : \mathbb{Q}[b_1, \ldots, b_n, x] \to H^*_{S^1}(U_S)$ denote the composition of the Kirwan map with restriction to U_S. Our claim is that $\ker \phi = \mathcal{J}_S$. For every short subset T containing S, let

$$\phi_T : \mathbb{Q}[b_1, \ldots, b_n, x] \to H^*_{S^1}(X_T \cap U_S)$$

denote the composition of the Kirwan map with restriction to $X_T \cap U_S$, and let

$$J_T = \ker \phi_T.$$

Similarly, let

$$\phi_\emptyset : \mathbb{Q}[b_1, \ldots, b_n, x] \to H^*_{S^1}(M_S)$$

be the natural map, and let

$$J_\emptyset = \ker \phi_\emptyset.$$

The kernel of the restriction map $H^*_{S^1}(U_S) \to H^*_{S^1}(U_S^{S^1})$ to the fixed point set of U_S is a torsion module over $H^*_{S^1}(pt)$ [AB, 3.5], and Proposition 3.5 tells us that $H^*_{S^1}(U_S)$ is a free $H^*_{S^1}(pt)$-module. Hence the restriction map is injective, and we have

$$\ker \phi = \ker \phi_\emptyset \cap \bigcap_{T \supseteq S} \ker \phi_T.$$

We know that $X_T \cap U_S \cong \mathbb{C}P^{|S|-2}$ for all short $T \supseteq S$, therefore

$$H^*_{S^1}(X_T \cap U_S) \cong \mathbb{Q}[h, x]/h^{|S|-1}.$$

Furthermore, we know that for all $i \in T$, the restriction of $b_i + x$ to $H^*_{S^1}(U_T)$ is represented by the divisor $W_i \cap U_T$ (see Remark 4.3), and therefore restricts to the class of a hyperplane on $X_T \cap U_S$. Hence $\phi_T(b_i + x) = h$ for all $i \in T$. On the other hand, for $j \in T^c$, the class b_j

23

is represented by the divisor Z_{1j} on X, which is disjoint from $X_T \cap U_S$, hence $\phi(b_j) = 0$ for all $j \in T^c$. Thus we conclude that

$$\ker \phi_T = \left\langle b_1 - b_i, \; b_j, (b_1 + x)^{|S|-1} \mid i \in T, j \in T^c \right\rangle.$$

Lemma 4.4 *The intersection* $\bigcap_{T \supseteq S} \ker \phi_T$ *is equal to*

$$\left\langle b_1 - b_i, \; b_j(b_1 - b_j), \prod_{j \in R} b_j, (b_1 + x)^{|S|-1} \;\middle|\; i \in S, j \in S^c, R \cup S \; long \right\rangle.$$

Proof: First, since the variable x appears only in the generator $(b_1 + x)^{|S|-1}$, which is contained in every ideal on both sides of the equation, we may reduce the problem to showing that

$$\bigcap_{T \supseteq S} \left\langle b_1 - b_i, \; b_j \mid i \in T, j \in T^c \right\rangle = \left\langle b_1 - b_i, \; b_j(b_1 - b_j), \prod_{j \in R} b_j \;\middle|\; i \in S, j \in S^c, R \cup S \; long \right\rangle \tag{3}$$

in the ring $\mathbb{Q}[b_1, \ldots, b_n]$. Both ideals cut out the (reducible) variety

$$\bigcup_{T \supseteq S} Y_T \subseteq \operatorname{Spec} \mathbb{Q}[b_1, \ldots, b_n],$$

where

$$Y_T = \left\{ (z_1, \ldots, z_n) \;\middle|\; z_i = z_1 \; \forall i \in S, z_j = 0 \; \forall j \in S^c \right\}.$$

The left hand side of Equation (3) is an intersection of prime ideals, and is therefore radical. Thus by Hilbert's Nullstellensatz, it is sufficient to prove that the right hand side of Equation (3) is radical. This involves showing that the ideal is saturated, with Hilbert polynomial equal to the constant $\#\{\text{short } T \supseteq S\}$.

The degree k piece of the quotient

$$\mathbb{Q}[b_1, \ldots, b_n] / \left\langle b_1 - b_i, \; b_j(b_1 - b_j) \mid i \in S, j \in S^c \right\rangle$$

has a basis of elements of the form

$$b_1^{e_1} \prod_{j \in S^c} b_j^{e_j},$$

where $e_j \in \{0, 1\}$ for all $j > 0$, and $e_1 + \sum_{j \in S^c} e_j = k$. The subset of these elements with the property that $S \cup \{j \mid e_j = 1\}$ is short descends to a basis for the degree k part of the ring

$$\mathbb{Q}[b_1, \ldots, b_n] \Big/ \left\langle b_1 - b_i, \; b_j(b_1 - b_j), \prod_{j \in R} b_j \;\middle|\; i \in S, j \in S^c, R \cup S \; long \right\rangle,$$

hence our ideal has the desired Hilbert polynomial. It is also clear from this description that if an element a of the quotient ring is nonzero, so is $b_1^d \cdot a$ for any $d \geq 0$, hence our ideal is saturated. $\square$

It now remains to show that

$$\mathcal{J}_S = \left\langle b_1 - b_i, \ b_j(b_1 - b_j), \prod_{j \in R} b_j, \ (b_1 + x)^{|S|-1} \ \middle| \ i \in S, j \in S^c, R \cup S \text{ long} \right\rangle \cap \ker \phi_\emptyset.$$

The fact that $\mathcal{J}_S$ is contained in the intersection is clear. To show the opposite containment, consider an element

$$a + \eta \cdot (b_1 + x)^{|S|-1} \in \left\langle b_1 - b_i, \ b_j(b_1 - b_j), \prod_{j \in R} b_j, \ (b_1 + x)^{|S|-1} \ \middle| \ i \in S, j \in S^c, R \cup S \text{ long} \right\rangle,$$

with

$$a \in \left\langle b_1 - b_i, \ b_j(b_1 - b_j), \prod_{j \in R} b_j \ \middle| \ i \in S, j \in S^c, R \cup S \text{ long} \right\rangle,$$

and suppose that we also have

$$a + \eta \cdot (b_1 + x)^{|S|-1} \in \ker \phi_\emptyset.$$

Lemma 4.5 [HK2] *The kernel of $\phi_\emptyset$ is equal to*

$$\left\langle b_1 - b_i, \ b_j(b_1 - b_j), \prod_{j \in R} b_j, \ (b_1 + x)^{|S|-1} b_1^{-1} \left(\prod_{j \in L}(b_j - b_1) - \prod_{j \in L} b_j \right) \right\rangle,$$

where $i \in S, j \in S^c$, and $R, L \subseteq S^c$, with $R \cup S$ and L both long.

Lemma 4.5 tells us that $a \in \ker \phi_\emptyset$, therefore

$$\eta \cdot (b_1 + x)^{|S|-1} \in \ker \phi_\emptyset.$$

But $(b_1 + x)^{|S|-1}$ is represented in $H_{S^1}^*(U_S)$ by the subvariety M_S (see Remark 4.3), hence

$$0 = \phi_\emptyset(\eta \cdot (b_1 + x)^{|S|-1}) = \phi_\emptyset(\eta) \cdot e(M_S),$$

where $e(M_S)$ is the equivariant Euler class of the normal bundle to M_S inside of U_S. Since the equivariant Euler class of the normal bundle to a component of the fixed point set is never a zero-divisor, we have $\eta \in \ker \phi_\emptyset$. Then by Equation 4.5,

$$a + \eta \cdot (b_1 + x)^{|S|-1} \in \mathcal{J}_S.$$

25

This completes the proof of Theorem 4.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 4.6 For arbitrary n and α, suppose that S is a maximal short subset. Then Corollary 4.2 tells us that $H^*(U_S) \cong \mathbb{Q}[b_1]/\langle b_1^{n-2} \rangle$. We conjecture that in this case we in fact have $U_S \cong \mathbb{C}P^{n-3}$.

Example 4.7 Consider the core component pictured in Example 2.3. By Theorem 4.2 and Remark 4.3,

$$H^*(U_S) \cong \mathbb{Q}[b_1, b_3, b_4, b_5] \Big/ \left\langle \begin{array}{c} b_3(b_1 - b_3),\ b_4(b_1 - b_4),\ b_5(b_1 - b_5),\ b_3 b_4,\ b_3 b_5,\ b_4 b_5, \\ b_1(b_1 - b_3 - b_4),\ b_1(b_1 - b_3 - b_5),\ b_1(b_1 - b_4 - b_5) \end{array} \right\rangle,$$

where b_1 is the fundamental class of M_S, and b_3, b_4, and b_5 are the negatives of the fundamental classes of the curves labeled 123, 124, and 125, respectively. Because the transverse intersection of two complex varieties is positive, we know that $-b_1 b_3[U_S] = 1$. With respect to the basis

$$\{b_1 - b_3 - b_4 - b_5,\ b_3,\ b_4,\ b_5\},$$

the intersection form on $H^2(U_S)$ is represented by the matrix

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Hence U_S is homeomorphic to the blow-up of $\mathbb{C}P^2$ at three points.

Example 4.8 Using the same $\alpha = 0 \oplus (1, 1, 3, 3, 3)$, consider the short subset $S = \{1, 3\}$. In this case, Theorem 4.2 tells us that

$$H^*(U_S) \cong \mathbb{Q}[b_1, b_2]/\left\langle b_1^2, b_2(b_1 - b_2) \right\rangle.$$

With respect to the basis $\{b_1 - b_2,\ b_2\}$, the intersection form on $H^2(U_S)$ is represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, hence U_S is homeomorphic to the blow-up of $\mathbb{C}P^2$ at a single point.

References

[AB] M. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology* 23 (1984) no. 1, 1–28.

[BD] R. Bielawski and A. Dancer. The geometry and topology of toric hyperkähler manifolds. *Comm. Anal. Geom.* 8 (2000), 727–760.

[HP] M. Harada and N. Proudfoot. Properties of the residual circle action on a hypertoric variety. To appear in *Pacific Journal of Mathematics*.

[HS] T. Hausel and B. Sturmfels. Toric hyperkähler varieties. math.AG/0203096.

[HK1] J-C. Hausmann, A. Knutson. Polygon spaces and Grassmannians. *L'Enseignement Mathématique*. 43 (1997), 173-198.

[HK2] J-C. Hausmann, A. Knutson. The cohomology ring of polygon spaces. *Ann. Inst. Fourier, Grenoble* 48 (1998), no. 1, 281-321.

[Ki] F. Kirwan. *Cohomology of quotients in symplectic and algebraic geometry.* Mathematical Notes 31, Princeton University Press, 1984.

[Kl] A. Klyachko. Spatial polygons and stable configurations of points in the projective line. Algebraic geometry and its applications (Yaroslavl, 1992), *Aspects Math.*, Vieweg, Braunschweig (1994), 67–84.

[K1] H. Konno. Cohomology rings of toric hyperkhler manifolds. *Internat. J. Math.* 11 (2000), no. 8, 1001–1026.

[K2] H. Konno. On the cohomology ring of the HyperKähler analogue of the Polygon Spaces. Integrable systems, topology, and physics (Tokyo, 2000), 129–149, *Contemp. Math.*, 309, Amer. Math. Soc., Providence, RI, 2002.

[N1] H. Nakajima. Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras. *Duke Mathematical Journal* 72 (1994) no. 2, 365–416.

[N2] H. Nakajima. Varieties associated with quivers. *Canadian Math. Soc. Conf. Proc.* 19 (1996), 139–157.

[N3] H. Nakajima. Quiver varieties and Kac-Moody algebras. *Duke Math. J.* 91 (1998), no. 3, 515–560.

[N4] H. Nakajima. Quiver varieties and finite dimensional representations of quantum affine algebras. *J. Amer. Math. Soc.* 14 (2001), no. 1, 145–238 (electronic).